

States on timelike hypersurfaces in quantum field theory

Robert Oeckl*

Instituto de Matemáticas, UNAM Campus Morelia, C.P. 58190, Morelia, Michoacán, Mexico

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We investigate the possibility of defining states on timelike hypersurfaces in quantum field theory. To this end we consider hyperplanes in the real massive Klein-Gordon theory using the Schrödinger representation. We find a well defined vacuum wave functional, existing on any hyperplane, with the remarkable property that it changes smoothly even under Euclidean rotation through the light-cone. Moreover, particles on timelike hyperplanes exist and occur in two variants, incoming and outgoing, distinguished by the sign of the energy. Multi-particle wave functionals take a form similar to those on spacelike hypersurfaces. The role of unitarity and the inner product is discussed.

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Traditionally, Hilbert spaces of states in quantum field theory are associated with spacelike hypersurfaces. This is rooted in quantization prescriptions relying on the initial value problem, i.e., a correspondence between solutions of the equations of motion and initial data on a spacelike hypersurface. Furthermore, it may seem that causality requires hypersurfaces carrying states to be spacelike for a probability interpretation via the inner product to make sense.

On the other hand, quantum field theory shows properties suggesting that this restriction to spacelike hypersurfaces is artificial. In particular, the crossing symmetry of transition amplitudes may be taken as a strong hint that state spaces associated with more general hypersurfaces should make sense [1]. One may push this line of reasoning further to arrive at an axiomatic formulation of quantum field theory in the spirit of topological quantum field theory [2]. This view may also be motivated from quantum gravity [1, 2, 3].

We limit ourselves here to the investigation of states on hypersurfaces that are hyperplanes and consider the real massive Klein-Gordon theory in Minkowski space. Since we need to be free from the constraints of a canonical approach we employ the Schrödinger representation of quantum field theory [4, 5, 6] together with Feynman's path integral [7].

SCHRÖDINGER REPRESENTATION

The classical massive Klein-Gordon theory is defined by the wave equation of motion $(\partial_0^2 - \sum_{i \geq 1} \partial_i^2 + m^2)\phi = 0$ for a real scalar field $\phi(x)$. States in the quantum theory are wave functionals, i.e., functions on field configurations at a given time. The Schrödinger equation is formally solved by Feynman's path integral. Thus, the time-evolution from a state Ψ at time t to a state Ψ' at time t' is expressible as

$$\Psi'(\varphi') = \int \mathcal{D}\varphi \Psi(\varphi) \int_{\phi|_t=\varphi, \phi|_{t'}=\varphi'} \mathcal{D}\phi e^{iS(\phi)}. \quad (1)$$

The outer integral is over field configurations φ at time t . The inner integral, called *field propagator* and denoted by $Z(\varphi, \varphi')$, is over space-time field configurations ϕ in the time interval $[t, t']$ which match the configurations φ and φ' . Since the action S is quadratic, the stationary phase approximation to this integral is exact. Using this fact, rewriting the action as a boundary integral and using a decomposition of classical solutions into positive and negative energy components yields the propagator

$$Z(\varphi, \varphi') = N \exp \left(-\frac{1}{2} \int d^3x (\varphi \quad \varphi') W \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \right). \quad (2)$$

The operator-valued matrix W is given by

$$W = \frac{-i\omega}{\sin \omega \Delta} \begin{pmatrix} \cos \omega \Delta & -1 \\ -1 & \cos \omega \Delta \end{pmatrix},$$

where $\Delta = t' - t$ and $\omega = \sqrt{-\sum_{i \geq 1} \partial_i^2 + m^2}$.

The vacuum state is characterized by its invariance under time-evolution. We make the ansatz

$$\Psi_0(\varphi) = C \exp \left(-\frac{1}{2} \int d^3x \varphi(x) (A\varphi)(x) \right) \quad (3)$$

for an unknown operator A . Evolving the state using the propagator (2) we find that invariance is equivalent to the equation $A^2 = \omega^2$. We make the conventional choice $A = \omega$. A one-particle state of momentum p is given by

$$\Psi_p(\varphi) = \check{\varphi}(p) \Psi_0(\varphi), \quad (4)$$

where $\check{\varphi}$ is the Fourier transform $\check{\varphi}(p) = 2E \int d^3x e^{ipx} \varphi(x)$.

VACUUM ON BOOSTED HYPERPLANES

We shall now be interested in states on spacelike hyperplanes that are not normal to the time axis. Since the effect of spatial rotations is essentially trivial it suffices to single out the x_1 -coordinate and consider hyperplanes whose normal vector lies in the plane spanned by

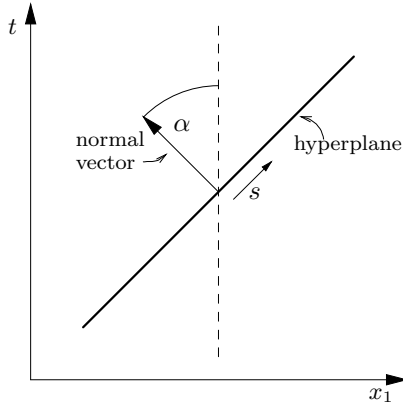


FIG. 1: Position of the hyperplane in (t, x_1) -coordinates. The s -coordinate runs along the hyperplane.

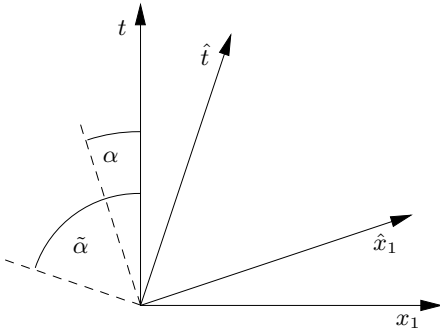


FIG. 2: Original (t, x_1) and boosted (\hat{t}, \hat{x}_1) coordinates.

x_1 and t , see Figure 1. We shall express everything in terms of *Euclidean* coordinates and angles with respect to the original coordinate system. Thus, call α the angle between the hyperplane and the time axis. Call s the rotated version of the x_1 -coordinate. Thus, within the hyperplane, $x_1 = s \cos \alpha$ and $t = s \sin \alpha$. The hyperplane is spacelike if $\alpha < \pi/4$.

Since the theory is manifestly Lorentz covariant we can produce states on these spacelike hyperplanes from those on the standard hyperplane by Lorentz boosts. Consider a boost in the (t, x_1) -plane with parameter γ . The boosted coordinates \hat{t}, \hat{x}_1 are given in terms of the original ones by

$$\begin{pmatrix} \hat{t} \\ \hat{x}_1 \end{pmatrix} = \begin{pmatrix} \cosh \gamma & -\sinh \gamma \\ -\sinh \gamma & \cosh \gamma \end{pmatrix} \begin{pmatrix} t \\ x_1 \end{pmatrix}, \quad (5)$$

see Figure 2. We shall consider the hyperplane spanned by (\hat{x}_1, x_2, x_3) . To bring this into accordance with the previous picture, the angles α in Figures 1 and 2 have to agree. The relation between \hat{x}_1 and s is then given by the length contraction factor $\rho = 1/\sqrt{\cosh 2\gamma}$ via $\hat{x}_1 = \rho s$. The Euclidean angle α is related to ρ via $\cos 2\alpha = \rho^2$.

Consider now the vacuum state on the hyperplane spanned by (s, x_2, x_3) . In terms of the boosted coordinates (\hat{x}_1, x_2, x_3) , it takes the usual form given by (3)

with $A = \hat{\omega}$ (we use a hat to indicate that the coordinate \hat{x}_1 instead of x_1 appears in ω). Transforming the expression to (s, x_2, x_3) -coordinates affects both the operator $\hat{\omega}$ as well as the integral measure $d\hat{x}_1$. Explicitly, $d\hat{x}_1 = \rho ds$ and $\hat{\omega} = \sqrt{-\rho^{-2}\partial_s^2 - \sum_{i \geq 2} \partial_i^2 + m^2}$, where ∂_s is the derivative in the s -coordinate. Abbreviating (x_2, x_3) collectively by \tilde{x} , we can write the resulting expression for the vacuum as

$$\Psi_0(\varphi) = C \exp \left(-\frac{1}{2} \int ds d^2 \tilde{x} \varphi(s, \tilde{x}) (\tau \varphi)(s, \tilde{x}) \right), \quad (6)$$

where $\tau = \sqrt{-\partial_s^2 + \cos 2\alpha (-\sum_{i \geq 2} \partial_i^2 + m^2)}$.

One might be surprised that the resulting expression takes a simple form in terms of *Euclidean* coordinates and angles. What is more, the vacuum functional appears to behave smoothly in the limit $\alpha \rightarrow \pi/4$ and even beyond the light-cone. Is it meaningful there?

VACUUM ON TIMELIKE HYPERPLANES

Note that we could have used an alternative route to obtain a general expression for the vacuum functional. Namely, for a given hyperplane we could have worked out the field propagator to a parallel hyperplane and then imposed invariance under the propagation using the analogue of the ansatz (3). The result would have been identical to (6) due to the Lorentz covariance of our setup. On the other hand, this suggests a characterization of the vacuum beyond the spacelike case as being invariant under propagation to a parallel hyperplane.

As for spacelike hyperplanes it suffices to start with a given one and obtain any other one through a Lorentz transformation. Thus, we single out the hyperplane spanned by the coordinates (t, x_2, x_3) . We wish to define states in analogy with the spacelike case as functions on field configurations on the hyperplane. From a quantization point of view the configuration space should be “half” of the phase-space, which in turn corresponds to the space of classical solutions. This means in particular that a configuration must be extendible to a classical solution. In the spacelike hyperplane case this is trivial. Any (reasonable) scalar function in space extends to a solution in space-time. For a timelike hyperplane this is not so. We need to restrict to configurations that do extend to classical solutions, calling them *physical* configurations.

Next, we define “evolution” in an interval $[x_1, x'_1]$ by the path integral over the space-time region defined by this interval, in analogy to (1). To evaluate the path integral we proceed in a manner analogous to the spacelike case [8]. The role of positive and negative energy contributions is now played by contributions with positive and negative sign of the momentum component in the

x_1 -direction. We arrive at the field propagator

$$Z(\varphi, \varphi') = N \exp \left(-\frac{1}{2} \int dt d^2 \tilde{x} (\varphi \varphi') W \left(\frac{\varphi}{\varphi'} \right) \right). \quad (7)$$

The operator-valued matrix W is given by

$$W = \frac{i\kappa_1}{\sin \kappa_1 \Delta} \begin{pmatrix} \cos \kappa_1 \Delta & -1 \\ -1 & \cos \kappa_1 \Delta \end{pmatrix},$$

where $\Delta = |x'_1 - x_1|$ and $\kappa_1 = \sqrt{-\partial_0^2 + \sum_{j \geq 2} \partial_j^2 - m^2}$.

We turn to the computation of the vacuum state. As already mentioned, we take as the characterizing condition the invariance under propagation between parallel hyperplanes. Making the ansatz (3) with respect to the (t, x_2, x_3) -hyperplane yields the condition $A^2 = \kappa_1^2$, similarly to the spacelike case. We set $A = \kappa_1$ and justify our choice of sign in a moment.

Given the candidate vacuum state we have defined we proceed to generalize it to arbitrary timelike hyperplanes by using Lorentz transformations. Without loss of generality we can restrict again to Lorentz boosts in the (t, x_1) -plane. Thus, consider the boost with parameter γ transforming coordinates via (5), recall Figure 2. Again we express everything in terms of Euclidean coordinates and angles according to Figure 1. Since now the boosted hyperplane is spanned by (\hat{t}, x_2, x_3) we find that the angle α in Figure 1 corresponds to the angle $\tilde{\alpha}$ in Figure 2. The coordinate s is related to \hat{t} by the time dilation factor ρ (the same ρ as above) via $\hat{t} = \rho s$. The relation between α and ρ now takes the form $\cos 2\alpha = -\rho^2$. Transforming the operator κ_1 as well as the integration measure in the expression for the vacuum functional, we obtain precisely formula (6) again. However, this time the range for α is $\pi/4 < \alpha \leq \pi/2$. The agreement with the spacelike case in the limit $\alpha \rightarrow \pi/4$ fixes the sign ambiguity in the choice of A .

PARTICLES ON TIMELIKE HYPERPLANES

While the expression for a vacuum state found above is very suggestive, a convincing justification that states on timelike hyperplanes are sensible requires us to consider particle states. We may expect crucial differences between such particle states and those on spacelike hypersurfaces. It will be sufficient to consider particle states on the hyperplane spanned by (t, x_2, x_3) .

Since a state is a function on physical configurations on the hyperplane, a basis of one-particle states may be characterized by the Fourier modes in this hyperplane. In the standard spacelike case these are labeled by 3-momentum. In the present timelike case these are labeled by the energy and the momentum in the x_2 - and x_3 -directions. The restriction of the energy to satisfy $E^2 \geq \tilde{p}^2 + m^2$ corresponds precisely to the restriction for configurations to physical ones.

Comparison with the spacelike situation (4) suggests that a one-particle state should be described by the functional

$$\Psi_{E, \tilde{p}}^\pm(\varphi) = \tilde{\varphi}^\pm(E, \tilde{p}) \Psi_0(\varphi), \quad (8)$$

where now $\tilde{\varphi}$ is the Fourier transform in the hyperplane,

$$\tilde{\varphi}^\pm(E, \tilde{p}) = 2p_1 \int dt d^2 \tilde{x} e^{\pm i(Et - \tilde{p}\tilde{x})} \varphi(t, \tilde{x}).$$

We use here the convention that $E \geq 0$ and the actual sign of the energy is encoded in the extra index \pm . Indeed, one can check that the state (8) is an eigenstate under propagation between parallel hyperplanes. Its eigenvalue is $e^{ip_1 \Delta}$ where $\Delta = |x'_1 - x_1|$ and p_1 is the positive square root of $E^2 - \tilde{p}^2 - m^2$.

What is the interpretation of such states? In the spacelike case causality prescribes that a state must be purely incoming or outgoing depending on whether it lies at the beginning or the end of a (time-)evolution process. An analogue of this does not hold in the timelike case. Any individual particle might now be either incoming or outgoing. This choice is reflected in (8) by the index \pm , representing the sign of the energy value. To agree with the spacelike case (concerning the sign of the momentum components p_2 and p_3) we identify Ψ^- as an in-particle state and Ψ^+ as an out-particle state.

In spite of this apparent extra choice in the timelike case, the degrees of freedom of a particle are identical to those in the spacelike case. The reason is that the “missing” sign of the momentum component p_1 in the timelike case must be correlated with the sign of the energy. Namely, the momentum of an in-particle needs to point from the hyperplane into the propagation region and that of an out-particle outward. Thus, in both the space- and the timelike case we may characterize a particle by its 3-momentum.

Complex conjugation converts an in-particle state $\Psi_{E, \tilde{p}}^-$ to an out-particle state $\Psi_{E, \tilde{p}}^+$ and vice versa. Moreover, if we put the complex conjugate of a state on the opposite hyperplane (i.e., on the other side of the evolution region), then it describes the *same* state in terms of the particle 3-momentum. This also parallels the situation in the spacelike case, although it is more implicit there. Namely, consider a transition amplitude from a 1-particle state with momentum p to a 1-particle state of momentum p' ,

$$\langle \Psi_{p'} | U(\Delta) | \Psi_p \rangle = \int \mathcal{D}\varphi \mathcal{D}\varphi' \Psi_p(\varphi) \overline{\Psi_{p'}(\varphi')} Z(\varphi, \varphi'). \quad (9)$$

The usual interpretation of course is that the complex conjugation comes from the inner product, as we are pairing a bra- with a ket-state. However, we could equally well say that a change of orientation of the hyperplane on which the state lives requires a complex conjugation to describe the original state. That is, while on the

in-oriented hyperplane a 1-particle state of momentum p' is described by the wave functional Ψ_p , on the out-oriented hyperplane it is described by the wave functional $\bar{\Psi}_{p'} = \Psi_{-p'}$. While this difference of interpretation seems of little consequence in the spacelike case, it becomes significant in the timelike case. Indeed, the bra-ket notation is inherently linked to an orientation of the time-axis chosen from the outset, i.e., we can distinguish between “earlier” and “later”. In space, on the other hand, we can continuously rotate any orientation into its opposite, so fixing an orientation from the outset makes no sense.

We turn to the dynamics in the form of transition amplitudes. Mindful of the inadequacy of the bra-ket notation we denote an amplitude between states Ψ and Ψ' by $[\Psi, \Psi']$. States are understood as specified with respect not only to the hyperplane they live in, but also with respect to its orientation relative to the propagation region. Thus, no explicit complex conjugation as in (9) appears. Concretely, a transition amplitude between states Ψ and Ψ' on parallel hyperplanes located at x_1 and x'_1 respectively takes the form

$$[\Psi, \Psi'] = \int \mathcal{D}\varphi \mathcal{D}\varphi' \Psi(\varphi) \Psi'(\varphi') Z(\varphi, \varphi').$$

Here the integrals are over physical configurations φ, φ' on the hyperplanes and $Z(\varphi, \varphi')$ is the propagator (7).

Inserting an in-particle of momentum p on the hyperplane at x_1 and an out-particle of momentum p' at x'_1 we find the amplitude to be

$$2p_1(2\pi)^3 \delta(E - E') \delta^2(\tilde{p} - \tilde{p}') e^{ip_1 \Delta}.$$

On the other hand, the transition amplitude between in-particles on both hyperplanes or out-particles on both hyperplanes is zero. Both results are exactly what should be expected. The probability (and hence the amplitude) of having two particles go into the propagation region and none coming out should be zero, and vice versa. In contrast, having one particle coming in and one going out should mean in the non-interacting theory we consider that they are the “same” particle, hence the delta functions. The latter situation parallels the spacelike case.

We finally turn to multi-particle states. As in the spacelike case, an n -particle wave functional is a certain polynomial of degree n in functionals $\tilde{\varphi}^\pm(E, \tilde{p})$ and delta functions, multiplied with the vacuum functional. Indeed, the structure of the polynomial is exactly the same as in the spacelike case (see [6] for the latter). One only has to take into account that the sign in the delta functions relating to one in- and one out-particle will be reversed due to our convention of “externalizing” the sign of the energy. Furthermore, delta functions relating to two in- or two out-particles may be dropped as they can never be satisfied.

A WORD ON UNITARITY

The failure of the bra-ket notation in the timelike case indicates that the role and interpretation of the inner product has to be considered carefully. Usually, an inner product and associated probability interpretation is implemented early on in the construction of a quantum mechanical model, before the dynamics. Consistency then requires that the dynamics be compatible with the inner product, i.e., be unitary.

The present context suggests a different route. We start by defining state spaces associated to hyperplanes, and transition amplitudes. States on the same hyperplane, but with different orientation belong a priori to different spaces. However, physically there is a correspondence between such states, i.e., we know whether a given in-particle is the “same” (has the same 3-momentum) as a given out-particle. We saw that this identification is given by the complex conjugation of the wave functional.

We can now demand that evolving a given state for 0 time duration (or space distance) yields the same state with probability one. The resulting expression yields the normalization condition for states, $\int \mathcal{D}\varphi \Psi(\varphi) \bar{\Psi}(\varphi) = 1$, where the integral is over *physical* configurations only. Indeed, this gives rise to an inner product and with it the possibility of evaluating the overlap of states etc. Unitarity is now hidden in the requirement that identification between states on opposite hyperplanes be preserved under propagation. Thus, we recover an essentially conventional probability interpretation. However, it is important to recognize that probabilities are in general *conditional* probabilities implicating the whole measurement process. More precisely, amplitudes give rise to probabilities for certain particles on *both* hyperplanes to be observed, conditional on certain other particles on *both* hyperplanes to be present (prepared).

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* Electronic address: robert@matmor.unam.mx

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